Estimation Methods with Algorithms for Robotics

Junseo Kim
Department of Mechanical and Industrial Engineering, Toronto Metropolitan University
Robotics and Computer Vision Lab (RCVL)

I. Linear Gaussian Estimation

Linear Gaussian Estimation is a technique used in signal processing and control systems. This document will discuss three different methods of estimation: Batch Least-Squares Estimation, Recursive Discrete-Time Filtering (also known as the Kalman Filter), and Batch Continuous-time Estimation (also known as Gaussian Process Regression) [**?**].

A. Recursive Discrete-Time Filtering (Kalman Filter)

1) Algorithm:

Prediction:

$$\tilde{x}_k = A_{k-1}\hat{x}_{k-1} + v_k$$
$$\tilde{P}_k = A_{k-1}\hat{P}_{k-1}A_{k-1}^T + Q_k$$

Correction:

$$K_k = \tilde{P}_k C_k^T (C_k \tilde{P}_k C_k^T + R_k)^{-1}$$
$$\hat{x}_k = \tilde{x}_k + K_k(y_k - C_k\tilde{x}_k)$$
$$\hat{P}_k = (I - K_k C_k)\tilde{P}_k$$

The system uses the previous state and the state transition to predict the current state and previous error covariance, state transition, and the process noise to predict current error covariance. Then, the algorithm calculates the Kalman gain with the current observation and updates the state with the observation model. Error covariance is also updated with the observation model and Kalman gain [**?**].

2) Key Differences: As this is a recursive filter, the estimation method does not require the entire history and is operated based on the current state and observation.

3) Advantages: This is optimal for linear systems and requires less memory than non-recursive filters. This method requires only current state and observation, which is computationally efficient.

4) Disadvantages: As this method assumes the system is linear and Gaussian, it may not be suitable for real-world application and can be sensitive to the initial state.

B. Batch Continuous-time Estimation (Gaussian Process Regression

1) Algorithm:

$$\hat{x}_{k-1}, \hat{P}_{k-1}$$

The goal of this algorithm is to compute $\hat{x}_k, \hat{P}_k$ with the data up to time k.

$$\hat{x}_{k-1}, \hat{P}_{k-1}, v_k, y_k \rightarrow \hat{x}_k, \hat{P}_k$$

The MAP solution is given by:

$$(H^T W^{-1} H)\hat{x} = H^T W^{-1} z$$

where H and W are defined as

$$z = \begin{bmatrix} \hat{x}_{k-1} \\ v_k \\ y_k \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ -A_{k-1} \\ 1 \\ C_k \end{bmatrix}, \quad W = \begin{bmatrix} \hat{P}_{k-1} \\ Q_k \\ R_k \end{bmatrix}$$

The estimate at time k is given by:

$$\hat{x}_k = \hat{x}_{k-1} + \Lambda_k(\hat{x}_k - \hat{x}_{k-1})$$

The posterior covariance is given by

$$\hat{P}_{k,k} = \hat{P}_{k-1,k-1} + \Lambda_k(\hat{P}_{k,k} - \hat{P}_{k-1,k-1})\Lambda_k^T$$

2) Key Differences: This method works with continuous data, which will provide more accurate estimates.

3) Advantages: This method can provide more accurate estimates and can handle non-linearities.

4) Disadvantages: This method may not be feasible for extensive data sets and can be computationally expensive.

II. Non-Linear Gaussian Estimation

Non-Linear Gaussian Estimation is critical when dealing with systems with non-linear behavior. The Extended Kalman Filter (EKF) linearizes an estimate of the current mean and covariance for non-linear systems. The Particle Filter (PF), also known as the Sequential Monte Carlo method, can be a robust approach to representing posterior density function with a set of random samples with weights to compute the estimates.

A. Extended Kalman Filter

1) Algorithm:

$$x_{k|k-1} = f(x_{k-1|k-1}, u_k)$$

$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k$$

where $x_{k|k-1}$ is the predicted state, $x_{k-1|k-1}$ is the previous state, u_k is the control input,$P_{k|k-1}$ is the predicted covariance, F_k is the Jacobian of f, and Q_k is the process noise covariance.

$$y_k = z_k - h(x_{k|k-1})$$

$$S_k = H_k P_{k|k-1} H_k^T + R_k$$

$$K_k = P_{k|k-1} H_k^T S_k^{-1}$$

$$x_{k|k} = x_{k|k-1} + K_k y_k$$

$$P_{k|k} = (I - K_k H_k) P_{k|k-1}$$

where y_k is the measurement residual, z_k is the actual measurement, h is the observation model, S_k is the residual covariance, R_k is the measurement noise covariance, K_k is the optimal Kalman gain, $x_{k|k}$ is the updated state estimate, and $P_{k|k}$ is the updated covariance estimate.

2) Key Differences: Extended Kalman Filter can handle non-linearities by linearizing the current estimate.

3) Advantages: This method can provide efficient recursive means to estimate the system's state in real-time and is optimal if the system and the measurement noise are Gaussian.

4) Disadvantages: The Extended Kalman Filter can diverge for severe non-linearities and depends on the initial state estimate. Moreover, this method uses first-order Taylor series expansion, which can result in inaccuracies if the function is highly non-linear.

B. Particle Filter

1) Algorithm:

Prediction:

$$\tilde{x}_t^{[i]} = f(x_{t-1}^{[i]}, u_t, w_t^{[i]})$$

where $x_t^{[i]}$ is the state of the particle at time t, f is the state transition function, u_t is the control input and w_t is the process noise

Update:

$$\tilde{w}_t^{[i]} = p(z_t | x_t^{[i]})$$

weight w_t is determined based on the likelihood of the observation given the state of the i-th particle. The resampling process is performed based on the weights of the particles, where particles with higher weights are more likely to be selected.

Update:

$$\tilde{\tilde{x}}_t = \sum_i w_t^{[i]} x_t^{[i]}$$

2) Key differences: Particle Filters represent the state distribution by a set of particles for multi-modal distributions.

3) Advantages: This method can handle non-linear and non-Gaussian systems. Moreover, it can represent multi-modal distributions

4) Disadvantages: This method requires a good proposal distribution for generating particles. The particle filter will perform poorly if the proposal distribution is not good.

III. Outliers

Outlier estimation is used for the data analysis and model fitting, where data may have discrepancies with erroneous measurements and noise.

RANSAC is an iterative method that is particularly robust. It works by randomly selecting a subset of data to fit the model well and repeating until the algorithm finds the best model.

M-estimation aims to minimize the influence of outliers on the estimated model parameters by using a weight function in the objective function to reduce the influence of data points with large residuals.

Covariance estimation can work with multivariate data with a covariance matrix. Outliers are typically identified as data points that have a large Mahalanobis distance.[1]

A. RANSAC

1) Algorithm: The homography matrix is defined based on the source points and destination points

$$\mathbf{A} = \begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -ux & -uy & -u \\ 0 & 0 & 0 & x & y & 1 & -vx & -vy & -v \end{bmatrix}$$

where the source points are in (x, y) and the destination points are in (u, v).

Once matrix A is constructed, Singular Value Decomposition (SVD) is used for the system. The projected points are then calculated using the homography matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \mathbf{H} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The Euclidean distance between the projected points and destination points are calculated:

$$\text{error} = \sqrt{(x - x')^2 + (y - y')^2}$$

2) Key Differences: This algorithm is designed to handle a large proportion of outliers in the data and works by repeatedly selecting a random subset of the data and fitting a model to the subset.

3) Advantages: RANSAC is robust to outliers and can provide accurate estimates of the model parameters.

4) Disadvantages: number of iterations required and the threshold for the inliers can significantly affect the performance. RANSAC can also be computationally intensive due to the large number of iterations.

References

[1] Giorgio Grisetti, Rainer Kümmerle, Cyrill Stachniss, and Wolfram Burgard. A tutorial on graph-based slam. *IEEE Intelligent Transportation Systems Magazine*, 2(4):31–43, 2010.